Journal of Global Optimization **12:** 405–413, 1998.

Numerical Methods for Solution of the Multiextremal Problems Connected with the Inverse Mathematical Programming Problems*

VALERIAN BULATOV
Institute of Dynamics System and Control Theory, and Institute of Energy Systems, Russian Academy of Sciences, Irkutsk, Russia

(Received 14 August 1995; accepted 2 September 1997)

Abstract. We consider a quite general concept of the inverse mathematical programming problem. A brief description of the connection between primal and inverse problems is given. We show that for some primal convex problems the inverse problem is a d.c. programming problem and we present a cutting plane method for solution of the latter one.

Key words: Concave minorant, Cutting planes, d.c. Functions, Inverse problems, Multiextremal problems.

1. Introduction

Many mathematical programming problems can naturally be associated with the so-called inverse problems. A pair of problems is called self-inverse problems if solution or a part of solution to one problem is presented in the description of another. Such a definition is of an arbitrary nature as to which problem should be considered as the primal one and which as the inverse. A more thoroughly studied problem is often referred to as primal.

Consider the following example. Let $m \times n$ matrix A and vectors $c \in E^n, b \in E^m$ be given. Find

$$x^* \in \text{Argmin}\{c^T x : x \in R\}, \tag{1}$$

$$R = \{x : Ax \leqslant b, x \geqslant 0\}, \tag{2}$$

here we assume that R is a compact set.

Associate the following problem with Problems (1) and (2). Let x^* be given. Find vectors $c^* \in R_c, b^* \in R_b$ such that

$$x^* = \arg\min_x \{{c^*}^T x : Ax \leqslant b^*, x \geqslant 0\}, \tag{3}$$

where R_c, R_b are convex compact sets.

* Supported by RFFI Grant No. 98-01-00338 and SNSF Contract No. 7IP.

It is natural to call the linear programming Problems (1) and (2) the primal problem and Problem (3) the inverse one. It is obvious to associate a family of inverse problems of kind (3) with Problem (1) and (2). It depends on what we consider as the varying parameter: elements of the matrix A, vector c or vector b.

2. Inverse mathematical programming problems

We use a more general definition of the inverse mathematical programming problem (see Antipin [1]). Let the following parametric family of mathematical programming problems be given:

$$\min_x\{\varphi(x,u) : g(x,u) \leqslant 0, x \in R_x\}, \tag{4}$$

where $\varphi(x,u)$ is a continuous function, $g(x,u)$ is a continuous vector function ($g \in E^{m_1}$), R_x is a compact set, $u \in E^m$ is a vectorial parameter, and $R_x \subset E^n$.

From Problem (4) it is necessary to find a pair x^*, u^* which satisfies the condition

$$(x^*,u^*) \in R_{x,u} = \{(x,u) : f(x,u) \leqslant 0, w(x,u) = 0, u \in R_u\}, \tag{5}$$

where $f \in E^{m_2}, w \in E^{m_3}$ are continuous vector functions, and $R_u \subset E^m$ is a compact set. In other words, find a pair x^*, u^*, such that

$$x^* \in \text{Argmin}\{\varphi(x,u^*), g(x,u^*) \leqslant 0, x \in R_x\}, \tag{6}$$

$$f(x^*,u^*) \leqslant 0, w(x^*,u^*) = 0, u^* \in R_u. \tag{7}$$

We associate this with the primal problem of the following type: find

$$(\overline{x},\overline{u}) \in \text{Argmin}\{\varphi(x,u), g(x,u) \leqslant 0, x \in R_x, (x,u) \in R_{x,u}\}. \tag{8}$$

with inverse Problems (6) and (7). It is not difficult to see that a pair (x^*,u^*) is feasible for Problem (8), hence,

$$\varphi(\overline{x},\overline{u}) \leqslant \varphi(x^*,u^*).$$

This inequality shows a connection between the primal and inverse problems.

If in (7) we have $f(x,u) \equiv 0, w(x,u) = x - x^* = 0, (m_3 = n), R_u \subset E^n$ and input vector x^*, then from (6), (7) we obtain a standard form of the inverse mathematical programming problem: find vector u^* such that

$$x^* \in \text{Argmin}\{\varphi(x,u^*) : g(x,u^*) \leqslant 0, x \in R_x\}. \tag{9}$$

From now on we assume that $\varphi(x,u)$ and vector functions $g(x,u), f(x,u)$ are convex in (x,u), $w(x,u)$ is an affine function and R_x, R_u are convex closed sets. Then primal Problem (8) is a convex programming problem. Let us investigate the complexity of inverse Problems (6) and (7). To do this we use another form of Problems (6) and (7). For any $u \in R_u$ denote

$$R(u) = \{x : g(x,u) \leqslant 0, x \in R_x\}, \tag{10}$$

$$\varphi_0(u) = \min_x\{\varphi(x,u) : x \in R(u)\}, \tag{11}$$

$$M(u) = \text{Argmin}\{\varphi(x,u) : x \in R(u)\}. \tag{12}$$

Assume that $R(u) \neq \emptyset, M(u) \neq \emptyset$. Due to the assumptions, $\varphi_0(u)$ is a convex function. Then we can interpret the initial inverse problem in the following way: find vector $u^* \in R_u$, such that in the optimal set $M(u^*)$ of Problem (12) the vector $x^* \in R_{x,u^*} = \{x : f(x,u^*) \leqslant 0, w(x,u^*) = 0\}$ exists.

In [2] the constraint $x \in M(u)$ is called the extremal type constraint. Some simple problems with the extremal type constraints were studied in [3]. These constraints can be given as the standard equalities and inequalities:

$$\varphi(x,u) \leqslant \varphi_0(u), \qquad g(x,u) \leqslant 0, \quad x \in R_x.$$

Then inverse Problems (6) and (7) can be rewritten in the following way: find a pair x^*, u^* which satisfies the system

$$\psi(x,u) = \varphi(x,u) - \varphi_0(u) \leqslant 0, \tag{13}$$

$$g(x,u) \leqslant 0, \qquad f(x,u) \leqslant 0, \qquad w(x,u) = 0, \quad x \in R_x, \quad u \in R_u. \tag{14}$$

Due to the convexity of sets R_x, R_u, and the functions $g(x,u), f(x,u)$ and linearity of $w(x,u)$, the Constraints (14) determine a convex set in E^{n+m}. However, the left-hand part of Inequality (13) is given by the difference of two convex functions. This complicates the solution of the inverse problem. Moreover, function $\varphi_0(u)$ is given implicitly and the set determined by Inequality (13) does not satisfy the regularity conditions since for any fixed $u \in R_u$ the set $\{\varphi(x,u) - \varphi_0(u) < 0, x \in R(u)\}$ is empty. This fact makes the solution of inverse mathematical problems even more complicated. At the same time primal Problem (8), under the assumptions made above, is a convex programming problem. All this confirms the statement that even if the primal problem is 'good', the inverse problem is almost always 'bad'.

Consider now a particular case of Problems (13) and (14). Assume that the functions $g(x,u), f(x,u)$ are linear, and the sets R_x, R_u are determined by a system of linear inequalities, i.e. Conditions (14) are given in the following form: $Ax + Bu \leqslant b$, where A and B are matrices of the corresponding size and b is a given vector. Then we associate the following implicit d.c. programming problem with Problems (13) and (14):

$$\min\{\psi(x,u) = \varphi(x,u) - \varphi_0(u) : (x,u) \in R\}, \tag{15}$$

where $R = \{(x,u) : Ax + Bu \leqslant b\}, \psi(x,u)$ is a continuously differentiable function. To solve Problem (15) we use the cutting plane method in E^{n+1} (see [4]).

Denote $y = (x,u,\alpha), \alpha \in E^1$. Assume that the epigraph of the function $\psi(x,u)$ over the feasible set R is embedded into the bounded form below polyhedron

$R^k = \{(x,u,\alpha) : D^k y \leqslant d^k, Ax + Bu \leqslant b\}$. Let $(x^k, u^k, \alpha_k) = y^k \in E^{m+n+1}$ be a solution of the linear programming problem:

$$\min\{\alpha : (x,u,\alpha) \in R^k\}. \tag{16}$$

A^k is a $(m+n+1) \times (m+n+1)$ matrix of constraints active at the point (x^k, u^k, α_k), i.e. $A^k y^k = b^k$, S^{kj} are columns of the inverse matrix $(A^k)^{-1}$. Write the equations of the rays originating from y^k to the adjacent vertices of the polyhedron R^k

$$y = y^k - \lambda^j S^{kj}, \qquad j = \overline{1, m+n+1}, \quad \lambda^j > 0. \tag{17}$$

Solve the convex programming problem

$$\min\{\varphi(x, u^k) : x \in R(u^k)\} = \varphi(\overline{x}^k, u^k) = \varphi_0(u^k). \tag{18}$$

Then find the points $\{y^{k,1}, \ldots, y^{k,n+m+1}\}$ of the intersections of rays (17) with the graph of the function

$$\begin{aligned} \psi_k(x,u) \; = \; & \varphi(x^k, u^k) + \nabla\varphi_x(x^k, u^k)^T (x - x^k) \\ & + \nabla\varphi_u(x^k, u^k)^T (u - u^k) - \varphi_0(\overline{x}u), \end{aligned}$$

i.e. with the surface $\psi_k(x,u) = \alpha$.

Draw a cutting plane $p^{k^T} y = \beta_k$ through these points, we determine the polyhedron

$$R^{k+1} = \{(x,u,\alpha) : (x,u,\alpha) \in R^k, p^{k^T} y \leqslant \beta_k\},$$

which does not contain (x^k, u^k, α_k). The next approximation $(x^{k+1}, u^{k+1}, \alpha_{k+1})$ is obtained from the linear programming problem

$$\min\{\alpha : (x,u,\alpha) \in R^{k+1}\}.$$

LEMMA 1. *(i)* $\psi_k(x,u)$ *is a concave function; (ii)* $\psi_k(x^k, u^k) = \varphi(x^k, u^k) - \varphi_0(u^k) = 0$; *(iii)* $\psi_k(x,u) \leqslant \varphi(x,u) - \varphi_0(u), \forall x \neq x^k, u \neq u^k$.

Proof. The first and second assertions are obvious. Let us prove the third assertion. From the definition of $\varphi_0(u)$ for any fixed $\overline{x}^k$ we have $\varphi_0(u) \leqslant \varphi(\overline{x}^k, u)$, i.e. $\varphi(x,u) - \varphi_0(u) \geqslant \varphi(x,u) - \varphi(\overline{x}^k, u)$. Then due to the convexity of $\varphi(x,u)$

$$\begin{aligned} & \varphi(x,u) - \varphi_0(u) \geqslant \varphi(x,u) - \varphi(\overline{x}^k, u) \geqslant \varphi(x^k, u^k) \\ & + \nabla\varphi_x(x^k, u^k)^T (x - x^k) + \nabla\varphi_u(x^k, u^k)^T (u - u^k) - \varphi(\overline{x}^k, u) = \psi_k(x,u). \end{aligned}$$

Hence, $\psi_k(x,u)$ is a concave minorant of the minimized function, so it can be used for constructing correct cuts in E^{m+n+1}. The proof of the algorithm convergence is given in [5]. □

3. The Huard's centres method in inverse mathematical programming problems

The method described in the previous section can be extended to the nonlinear constraints case in (14). Here we use the Huard's centres method in which constraints are moved into the minimand

$$\psi_k(x) = \max\{\varphi_0(x) - \varphi_0(x^k), \varphi_1(x), ..., \varphi_m(x)\}. \tag{19}$$

Then the mathematical programming problem

$$\min\{\varphi_0 : \varphi_i(x) \leqslant 0, i = \overline{1,m}, x \in R\}, \tag{20}$$

where R is a convex polytope, can be solved by the following iterative procedure

$$x^{k+1} = \operatorname{argmin}\{\psi_k(x) : x \in R\}. \tag{21}$$

Let us show the connection between the Huard's method [6] and the embeddings methods described in [3].

Let $\varphi_0^* = \min\{\varphi_0(x) : x \in R^0\}$, where $R^0 = \{x : x \in R, \varphi_i(x) \leqslant 0, i = \overline{1,m}\}$, then the solving of Problem (20) is equivalent to the solving of the following mathematical programming problem

$$\min\{\psi(x) : x \in R\}, \tag{22}$$

where

$$\psi(x) = \max\{\varphi_0(x) - \varphi_0^*, \varphi_1(x), ..., \varphi_m(x)\}. \tag{23}$$

Define the epigraph R_{n+1} of the function $\psi(x)$

$$R_{n+1} = \{(x, x_{n+1}) : x \in R, x_{n+1} \in E^1, \psi(x) \leqslant x_{n+1}\}.$$

Then Problems (22) and (23) is equivalent to the following one

$$\min\{x_{n+1} : (x, x_{n+1}) \in R_{n+1}\}. \tag{24}$$

In [3] the following definition of the embeddings, methods for solving Problem (24) are given.

Let the set $R_{n+1}^k \supset R_{n+1}$ be given and let the point (x^k, x_{n+1}^k) be an optimal solution of the problem

$$\min\{x_{n+1} : (x, x_{n+1}) \in R_{n+1}^k\}. \tag{25}$$

Construct a set $R_{n+1}^{k+1} \supset R_{n+1}$ such that $(x^k, x_{n+1}^k) \notin R_{n+1}^{k+1}$.
Find the next approximation (x^{k+1}, x_{n+1}^{k+1}) as a solution to the problem

$$\min\{x_{n+1} : (x, x_{n+1}) \in R_{n+1}^{k+1}\}. \tag{26}$$

If there exists $K_1 \subset K = \{1, 2, ...\}$, such that

$$\lim_{k\to\infty} x_{n+1}^k = \varphi_0^*, k \in K_1, \tag{27}$$

then Methods (26) and (27) are called methods of the objective epigraph embedding, or simply embedding methods.

THEOREM 1. *Let sets* $\{R_{n+1}^k\}$ *be determined in the following way:*

$$R_{n+1}^k = \{(x, x_{n+1}) : x \in R, \Phi_k(x) \leqslant x_{n+1}, k \in K\},$$

where $\{\Phi_k(x)\}$ *is a family of equicontinuous functions, such that*

$$\Phi_i(x^i) \leqslant \Phi_k(x^k), \forall k, i \in K (k > i), \tag{28}$$

$$\Phi_k(x^i) \geqslant \varphi_0^*, \forall k, i \in K (k > i), \tag{29}$$

where $x^i =$ *argmin*$\{x_{n+1} : (x, x_{n+1}) \in R_{n+1}^i\}, i \in K$. *Then there exists* $K_1 \subset K$, *such that*

$$\lim_{k\to\infty} x_{n+1}^k = \varphi_0^*, k \in K_1. \tag{30}$$

Proof. Choose from $\{x^k\}$ a convergent subsequence with numbers $k \in K_1 \subset K$. Due to (28) $\varphi_0^* \geqslant \Phi_k(x^k) \geqslant \Phi_i(x^i), \forall k, i \in K_1(k > i)$. Therefore, there exists

$$\lim_{k\to\infty} \Phi_k(x^k) = \lim_{k\to\infty} x_{n+1}^k = \varphi^1 \leqslant \varphi_0^*, \quad k \in K_1$$

or

$$\Phi_k(x^k) \leqslant \varphi^1 \leqslant \varphi_0^*, \quad \forall k \in K_1. \tag{31}$$

Assume that $\varphi^1 < \varphi_0^*$, i.e.

$$\varphi_0^* - \varphi^1 > \varepsilon > 0. \tag{32}$$

Substituting the left-hand parts of (29) and (31) in (32) we obtain a stronger inequality

$$\Phi_k(x^i) - \Phi_k(x^k) \geqslant \varepsilon, \quad \forall k > i, \quad k, i \in K_1.$$

Then, due to the equicontinuity of $\{\Phi_k(x)\}$, there exists $\delta > 0$, such that

$$\|x^i - x^k\| > \delta, \quad \forall k > i.$$

The latter contradicts the convergence of x^k, therefore

$$\lim_{k\to\infty} x_{n+1}^k = \varphi_0^*, k \in K_1. \qquad \square$$

Many iterative processes fit in the scheme of embeddings methods and sufficiently satisfy the convergence conditions mentioned in the theorem, for example, the Pijavskii method [7], cutting planes methods [8], and many others. Here we show that the Huard's centres method fits the described scheme.

THEOREM 2. *The iterative process (19),(21) for solving problem (20) is the embedding method for the solution of the equivalent problem (22)-(23).*

Proof. Consider Problems (22) and (23) in the form (24). Determine the set

$$R_{n+1}^k = \{(x, x_{n+1}) : x \in R, \psi_k(x) \leqslant x_{n+1}\}. \tag{33}$$

Then the iterative process (19), (21) can be rewritten in the form

$$x^{k+1} = \text{argmin}\{x_{n+1} : (x, x_{n+1}) \in R_{n+1}^k\}. \tag{34}$$

It is obvious that $R_{n+1}^k \supset R_{n+1}$, moreover, due to the construction,

$$(x^k, x_{n+1}^k) \notin R_{n+1}^{k+1}.$$

Now we have to prove that there exists an index set $K_1 \subset K$, such that

$$\lim_{k\to\infty} \psi_k(x^k) = \lim_{k\to\infty} x_{n+1}^k = 0, \quad k \in K_1. \tag{35}$$

The sequence $\{\psi_i(x^i)\}$ is monotonously nondecreasing and nonpositive, i.e.

$$\psi_i(x^i) \leqslant \psi_k(x^k) \leqslant 0, \quad \forall k, i \quad \in K, i < k. \tag{36}$$

On the other hand, from (21) we have

$$\psi_k(x^i) \geqslant 0, \quad \forall i, k \in K, \quad i < k. \tag{37}$$

Hence, there exists

$$\lim_{k\to\infty} \psi_k(x^k) = \lim_{k\to\infty} x_{n+1}^k = P \leqslant 0, \tag{38}$$

i.e. $\psi_k(x^k) \leqslant P \leqslant 0$. Assume that

$$-P \geqslant \varepsilon > 0. \tag{39}$$

Adding (38) to (40) we obtain

$$\psi_k(x^i) - \psi_k(x^k) > \varepsilon, \quad k > i, \tag{40}$$

and since

$$\psi_k(x^i) = \varphi_0(x^i) - \varphi_0(x^k), \qquad \psi_k(x^k) = \varphi_0(x^k) - \varphi_0(x^k),$$

then from (41) we have $| \varphi_0(x^i) - \varphi_0(x^k) | > \varepsilon$. Therefore, due to the continuity of $\varphi_0(x)$ there exists $\delta > 0$, such that $\|x^i - x^k\| > \delta, i > k, i, k \in K$. The latter contradicts the boundedness of R. □

Now consider again the use of the centres method (19), (21) for solving the inverse problems of the type (13), (14). If $\psi(x, u) = \varphi(x, u) - \varphi_0(u)$ is a d.c. function, $g(x, u), f(x, u)$ are convex functions, $w(x, u) = 0, R_x, R_y$ are convex polytopes, then the Huard's method has the following form

$$x^{k+1} = \text{argmin}\{\psi_k(x, u) : (x, u) \in R\}, \qquad R = R_x \times R_u,$$

$$\psi_k(x,u) = \max\{\psi(x,u) - \psi(x^k,u^k), g_1(x,u), \ldots, g_{m_1}(x,u), \\ f_1(x,u), \ldots, f_{m_2}(x,u)\}. \qquad (41)$$

If in (42) the maximum is attained at $\psi(x,u)-\psi(x^k,u^k)$ then the concave minorant can be determined by Lemma 1. Otherwise, if the maximum is some function $g_j(x,u)$ (or $f_i(x,u)$), then the concave minorant is determined by the linearization of this function.

In conclusion we consider following examples.

EXAMPLE 1. In Problem (9) let $n = 2, m_1 = 2, m = 1, x^* = (1,1)$,

$$\varphi(x,u) = x_1^2 + 2x_2^2 - ux_1 - 2ux_2, \qquad g_1(x,u) = x_1 + ux_2 - 8,$$

$$g_2(x,u) = ux_1 - x_2 - 12, \quad R_x = E_+^2 = \{x : x_1 \geqslant 0, x_2 \geqslant 0\}, \quad R_u = E_+^1.$$

Then Problem (15) can be rewritten in the form

$$\min\{\psi(x^*,u) = \varphi(x^*,u) - \varphi_0(u)\}, \quad 0 \leqslant u \leqslant 7,$$

where $\varphi(x^*,u) = 3 - 3u$. Setting the accuracy $\varepsilon = 10^{-3}$ and the initial value $u^0 = 0$ we obtain the optimal value $u^* = 2$ after 8 iterations of the cutting plane method.

EXAMPLE 2. In Problem (9) let $n = 2, m_1 = 2, m = 2, x^* = (1,1)$,

$$\varphi(x,u) = u_2x_1^2 + u_1x_2^2 - u_1x_1 - 2u_1x_2, \qquad g_1(x,u) = u_2x_1 + u_1x_2 - 8,$$

$$g_2(x,u) = u_1x_1 - u_2x_2 - 12, \qquad R_x = R_u = E_+^2.$$

Starting four times from different initial values of u^0 we obtained (in an average of 10 iterations) four different solutions: $(5.35, 2.65), (3, 1.5), (2.69, 1.38)$, and $(2, 1)$.

References

1. Antipin, A.S. (1992), Inverse Problems of Optimization: Statement of the Problem and Approaches to Solution, *Inverse Mathematical Programming Problems*, Computer Center of the Russian Academy of Sciences, Moscow (in Russian).
2. Levitin, E.S. (1992), Mathematical Programming Problems under Extremal Type Constraints, in *Inverse Mathematical Programming Problems*, Computer Center of the Russian Academy of Sciences, Moscow (in Russian).
3. Bulatov, V.P. (1977), *Methods of Embeddings in Optimization Problems*, Nauka, Novosibirsk (in Russian).
4. Bulatov, V.P. (1987), Methods for Solving Multiextremal Problems (global search), in B.A. Beltiukov and V.P. Bulatov (eds.), *Methods of Numerical Analysis and Optimization*, Nauka, Novosibirsk, pp. 133–157 (in Russian).
5. Bulatov, V.P. and Khamisov, O.V. (1992), The Branch and Bound Method with Cuts in E^{n+1} for Solving Concave Programming Problems, *Lecture Notes in Control and Information Sciences* 180, 273–281.

6. Huard, P. (1967), Resolution of Mathematical Programming Problems with Nonlinear Constraints by the Method of Centers, in *Nonlinear Programming*, North-Holland, Amsterdam, pp. 206–219.
7. Pijavskii, S. (1967), An Algorithm for Finding the Absolute Minimum of a function, *Theory of Optimal Solutions*, Volume 2, Kiev, IK AN USSR, pp. 13–24 (in Russian).
8. Kelley, J. (1960), The Cutting Plane Method for Solving Convex Programs, *Journal SIAM* 8, 703–712.